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# Variable Structure System Controllers: Is the Switching Surface Optimal?

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**Abstract**

This paper presents a viewpoint on switching surface design in variable structure control. It gives a brief historical perspective about optimal control. The issue of constructing a switching surface has been addressed from different points of views by a number of researchers. One aspect that is not emphasized enough is the quest for optimality. Can the switching surface be made optimal; and if so, in what sense? Solutions with practical interest for linear composite systems (sub-systems or structures) are provided withing the framework of Linear Quadratic Regulator (LQR) theory.

**Key Words:** Euler Lagrange Equations, Linear Quadratic Regulator (LQR), Matrix Algebraic Ricatti Equation (MARE), Sliding Surface, Optimal Control, Optimal Switching Surface.

## 1.Introduction

Variable Structure Control (VSC) is a well-established area of research within the field of control systems. The methodology and the key steps in designing such a controller are well known and mastered, both in single input single output systems (SISO) and multivariable input and output systems (MIMO). The different stages in the design include the switching surface and the controller. Throughout the development of this new area of controller design technic, researchers of different backgrounds and experiences, brought about a variety of approaches in constructing the so-called sliding surface according to their performance objectives. Some of these objectives emphasize either the decoupling of the states or stability in sliding mode. As stated by Wu-Chung Su; Sergey V. Drakunov and Ümit Özgüner (1996), the theory of designing linear switching surfaces for linear dynamics has been developed in great depth and completeness. This paper highlights the fact that different approaches attempt to design a sliding surface that is optimal in some ways. Often, the intend is rather understood than being openly stated. We present a brief historical perspective of calculus of variation followed by the LQR problem and its solution. A brief review of the key principles of Variable Structures Controller design is also provided. Then, we now show how this result is used in sliding surface design.

## 2. Historical Perspective of Optimal Control

Throughout history, the quest for optimality stems from our observation of mother nature. The so called '*Fermat principle*' of the shortest time path travelled by light through any medium is purported to be discovered by the French mathematician, Pierre de Fermat (1601-1665). It seems obvious that the work of Sir Isaac Newton (1643-1723) and Herr Gottfried Wilhelm Leibniz (1646-1716) on Calculus, laid the foundations and paved the way for Johann Bernoulli (1667-1748) to bring about his 1696 '*Brachistochrone Problem*'. Then followed a period of time that witnessed the blossom of the so called '*Least Action Principle*' in Physics, steered by Maupertuis (1698-1759) and the French-Italian Joseph-Louis Lagrange (1736-1813). That was the way the '*Brachistochrone Problem*' was solved by Lagrange and laid the foundations of the Calculus of Variations. This is the beginning of the quest for the best or optimal solution for any practical problem and indeed of optimal control. It is now clear that optimal control is closely related in its origins to the theory of calculus of variations, as stated in Beccera (2008).

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To the best of the author's knowledge, some of the earliest approaches to apply this concept in variable structure controller design are due to Utkin and Young (1979); Wu et al. (1996); Koshkouei and Zinober (1996). The approach is clearly inspired and motivated by optimal control and Linear Quadratic Regulator (LQR). The same trends is also followed by Jafarov (2008,2009), to come about some optimality concept in switching surface design.

### 2.1. The Euler-Lagrange Equation

In Calculus of Variations, **The Euler-Lagrange Equation** states the necessary and sufficient condition for optimality. Here, we give some commonly found statements in the literature.

Given a scalar function in equation (2.1). We are asked to find a function $y = f(x)$ in (2.1); so that the quantity $J$ in (2.2), is minimized.

$f \ : \ \mathbb{R} \to \mathbb{R}$

$x \ \to f(x) \ = y$

$$y = f(x) \text{ and } y' = \frac{df}{dx} \quad \ldots \quad (2.1)$$

$$J = \int_{x_A}^{x_B} F(y, y', x)dx \quad \ldots \quad (2.2)$$

The integrand $F(y, y', x)$ is historically named **Lagrangian** and the **Euler-Lagrange equation** in (2.3) is the necessary and sufficient condition to get the suitable $y$.

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \quad \ldots \quad (2.3)$$

In the *'Brachistochrone Problem'* from Johann Bernoulli (1696), $J$ is the minimum time for a bead (subject only to gravity) from position $A = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$ to reach point $B = \begin{pmatrix} x_B \\ y_B \end{pmatrix}$ along the curve $y = f(x)$. It is now common knowledge that that curve is a cycloid.

In classical physics, one generally deals with the type of equation (2.4). The **Lagrangian** is a scalar but there are many variables $q_i$ ; $i = 1,2, \ldots . n$. These are the generalized coordinates of the system in an $\mathbb{R}^n$ space. The interested reader is referred to Raymond (2024); Morin, D. (2007); Tatum, J. B (2024), and other related texts.

$$I = \int_{t_0}^{t_1} L(q_1, \dot{q}_1 \ ; \ q_2, \dot{q}_2 \ ; \ldots ; \ t)dt \quad \ldots \quad (2.4)$$

An **Euler-Lagrange equation** (2.5) must be written for each coordinate $q_i$.

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_i}\right) \quad \ldots \quad (2.5)$$

## 3. The Optimal Control and the LQR problem

As defined by Becerra (2008), Optimal Control is the process of determining control and state trajectories for a dynamic system over a period of time to minimize cost. The theory of Optimal Control is concerned with operating a dynamic system at minimum cost. The case where the system dynamics are described by a set of linear differential equations and the cost is described by a quadratic function is called the LQ problem. One of the main results in the theory is that the solution is provided by the linear–quadratic regulator (LQR, stabilizing) as a feedback controller $u(x, t)$.

The formulation of the LQR (and the Kalman filter) is due to Rudolf Kalman (1930-2016) in the 1960s, in Kalman (1960). This essentially refers to linear systems that are considered when dealing with optimality.

### 3.1. The LQR Problem

In this section, we present the LQR problem and its solution. Then, it will be shown that the solution of this problem gives the intuition for the design of a switching surface in VSC that is deemed to be optimal.

As described by Becerra (2008), a special case of optimal control problem which is of particular importance arises when the objective function, that is the integrand in equation (3.3) is a quadratic function of $x$ and $u$, and the dynamic equations are linear. The resulting feedback law in this case is known as the (LQR).

The following LTI SISO system in (3.1) is considered.

$$\dot{x} = Ax + Bu \qquad \dots \qquad (3.1)$$
$$y = Cx + Du$$

Find $u$ in equation (3.2), when $r = 0$, so that we have a regulator (set point is optimal stabilizing position); so as to minimize the performance index $J$ in (3.3) subject to (3.4).

$$u = -Kx + r \qquad \dots \qquad (3.2)$$

So, we see that the solution is already assumed to be in a form of state feedback control that will stabilize the closed – loop system as shown in figure Figure 1.

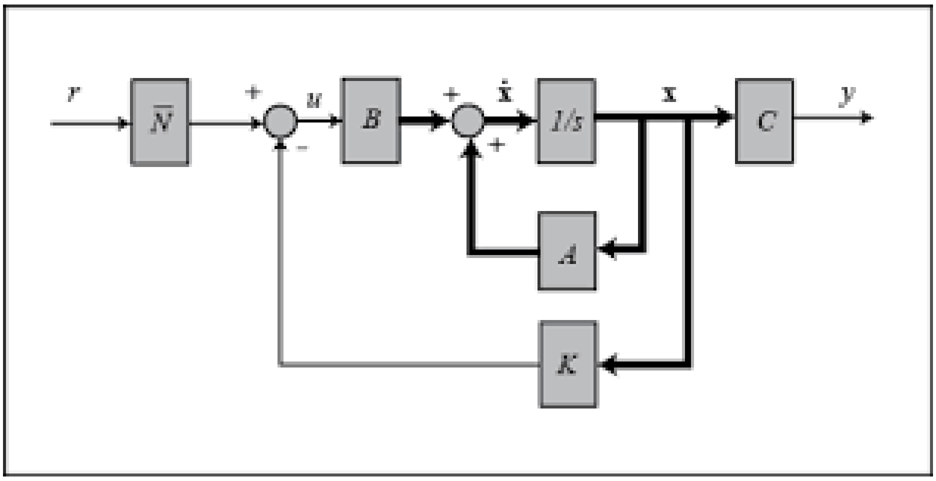


Fig.1: State Feedback of LTI System

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The extra block $\bar{\bar{N}}$, in which $r$ is fed represents a scaling factor that allows our system to track a reference step input at steady state.

$$J = \frac{1}{2}\int_0^{\infty}(x^T Qx + u^T Ru)dt \qquad \dots \qquad (3.3)$$
$$x = x(t) = (x_1, \quad \cdots \quad , x_n)^T \; ; \; \dot{x} = \frac{dx}{dt} \qquad \dots \qquad (3.4)$$
$$x_0 = x(t = 0) \; ; \; Q = Q^T \geq O \; ; \; P = P^T > O \quad \text{are known)} \qquad \dots \qquad (3.5)$$

### 3.2. Deriving The Solution to the LQR Problem

One can introduce the new Lagrangian as in (3.6) along with the Lagrange multipliers vector $\lambda^T$ and proceed by using the Euler-Lagrange equation as stated in section 2.1. Here, $J$ is replaced by $I$ as in equation (3.7). This is shown in (Eugenio Schuster, 2007).

$$L = \frac{1}{2}(x^T Qx + u^T Ru) + \lambda^T(Ax + Bu - \dot{x}) \qquad \dots \qquad (3.6)$$
$$I = \int_0^{\infty} Ldt \qquad \dots \qquad (3.7)$$

However, it is commonly made use of the so called '*Maximum Principle*', or 'P*ontryagin Maximum Principle*' that yields immediately the same result. The so-called Hamiltonian in (3.8) is then used for that purpose. This is shown in Murray (2006); Becerra (2008) and R. Bhattacharya (2023).

$$H = \frac{1}{2}(x^T Qx + u^T Ru) + \lambda^T(Ax + Bu) \qquad \dots \qquad (3.8)$$

In either case, one arrives at equations (3.9), (3.10) and (3.11) which are equivalent to solving the Matrix Algebraic Riccati Equation (MARE) in (3.12); for $P = P^T > O$.

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right) = Ax + Bu \; ; x_0 = x(t = 0) \qquad \dots \qquad (3.9)$$
$$-\dot{\lambda} = \left(\frac{\partial H}{\partial \lambda}\right) = Qx + A^T\lambda ; \quad \lambda = Px \qquad \dots \qquad (3.10)$$

$$0 = \left(\frac{\partial H}{\partial U}\right) = Bu + \lambda^T B \qquad \dots \qquad (3.11)$$

$$A^T P + PA - PBR^{-1}B^T P + Q = O \qquad \dots \qquad (3.12)$$

The stabilizing feedback control law $u$, solution to the LQR problem is given in (3.13).

$$u = -R^{-1}B^T Px = -Kx \qquad \dots \qquad (3.13)$$

$$J = \frac{1}{2}x^T(0)Px(0) \qquad \dots \qquad (3.14)$$

As shown in (Becerra Victor,2008), if the pair $(A, C)$ is observable, where $Q = C^T C,$ then the closed loop system $\dot{x} = (A - BK)x$ is asymptotically stable.

The linear quadratic regulator provides a way of stabilizing any linear system that is stabilizable. There are well established methods and software for solving the algebraic Ricatti equation in (3.12).

### 3.3. Application to Constructing Switching Surface

#### 3.3.1. A Brief Reminder of VSS

As stated in (Andrzej Bartoszewicz, 2022), the theory of Variable Structure Systems (VSS) with sliding modes is currently one of the most significant research topics within the control engineering domain. In this review, we refer only to SISO systems. In principle, VSS can be represented by the parallel connection of several different continuous subsystems (called *structures*) that act one at a time in the input-output path. A certain switching logic schedules in time the relevant structures, that can be either controlled or autonomous plants (Pisano,2000). A very important feature of VSS is order reduction, once the system is in sliding mode. This fact is largely highlighted in the literature as in Utkin V.I (1977); Utkin, V. I. (1983); Utkin,V. I. and Young, K.D. (1979). Zinober, A. S. I. (1981); Young, et al. (1999).

The design procedure consists of two steps. The first step is the choice of a manifold in the state space such that, once the state trajectory is constrained on it, the controlled plant exhibits the desired performance. The second step is the design of a discontinuous state-feedback control law, capable of forcing the system state to reach, in finite time, such a manifold accordingly called a sliding surface or switching surface.
In the SISO case, one surface (manifold) $G$ of dimension $1 \times n$ is constructed by using a normal vector $S$ to the state vector $x(t)$.

$$G = \{x | s(x) = Sx(t) = 0\} \qquad \dots \qquad (3.15)$$

$$S = [S_1 \quad S_2 \quad \dots \quad S_n] \qquad \dots \qquad (3.16)$$

Here, $S$ is the sliding surface matrix., It is a single row vector with $n$ components. This is a normal vector to $G.$

A switching surface that guarantees asymptotic stability is found by using Lyapunov's second method. Often, a Lyapunov function candidate $V(x, t)$ can be chosen as in equation (3.18), when dealing with a SISO linear system.

$V(x, t)$ must fulfil the requirements in (3.17) similarly to the energy function of the system.

i) $V(0, t) = 0$
ii) $V(x, t) \neq 0\,;\ x \neq 0$ … (3.17)
iii) $\dot{V}(s, x, t) = \frac{d}{dt}(V(x, t)) < 0$

A Lyapunov function candidate $V(x, t)$ is chosen as in (3.18);

$$V(x, t) = \frac{1}{2}s^T s \qquad \dots \qquad (3.18)$$

One can define a sliding /switching surface '*a priori*', as in (3.19).

$$s = s(x, t) = s(x) = Sx(t) = S_1x_1 + S_2x_2 + \cdots + S_nx_n = 0 \qquad \dots \qquad (3.19)$$

$$s = s(x, t) = S_1x_1 + S_2x_2 + \cdots + S_nx_n \qquad \dots \qquad (3.20)$$

Equation (3.20) above, is the scalar switching function.

$$\dot{V}(s, x, t) = \frac{d}{dt}(\frac{1}{2}s^T s) = \frac{1}{2}(s^T\dot{s} + \dot{s}^T s) = s^T\dot{s} < 0 \qquad \dots \qquad (3.21)$$

$s^T = s$ and $\dot{s}^T s = s^T\dot{s}$ ; (because $s$ is a scalar)

The switching/sliding surface in (3.19) guarantees asymptotic stability (stability in the sense of Lyapunov).

### 3.3.2. The Optimal Switching Surface

It is shown in (Kwon & Pearson,1977) and (Feliachi & Thowsen ,1981) that, stabilization of the linear time delay system with delays in the states by a memoryless feedback can be achieved if a control law $u_f(x,t)$ is chosen as in equation (3.22).

$$u_f(x,t) = -WB^T Px \quad \ldots \quad (3.22)$$

Where $P = P^T > O$ is either a solution to a Lyapunov equation as in (Wu et *al*., 1996) or a Riccati equation for delayed systems as in Jafarov (2009). In either case, the considerations that lead to establishing the Lyapunov or Ricatti equation are meant to meet some predefined design requirements.

$W$ is any matrix (of full rank) with appropriate dimensions.

$$S = WB^T P \quad \ldots \quad (3.23)$$

It is seen that the stabilizing Sliding Surface matrix $S$ is chosen to be the optimal feedback gain matrix $K$ that minimizes the performance index $J$ in (3.3) and (3.14).

$$S = R^{-1}B^T P = WB^T P = K \quad \ldots \quad (3.24)$$

Setting $W = R^{-1}$

The sliding surface designed in that way can be thought to be optimal.

## 4. Conclusion

In this paper, we tried to highlight the fact that, the concept of doing things in an optimal manner, is inherent to the control engineer mindset. Switching surface design in VSC makes no exception to that trends. Although such approach is often understood in most papers, it might be essential or critical to bear it in mind.

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**Appendix**

Given the vector valued functions $f, g$, as in (A1) and in (A2); we define the differentiation operations from 1. to 8.

$f, g \quad : \quad \mathbb{R}^n \quad \rightarrow \quad \mathbb{R}^m$

$x = (x_1 \quad x_2 \quad \dots \quad x_n)^T \rightarrow f(x) = (f_1(x) \quad f_2(x) \quad \dots \quad f_m(x))^T \quad \dots \quad$ (A1)

$x = (x_1 \quad x_2 \quad \dots \quad x_n)^T \rightarrow g(x) = (g_1(x) \quad g_2(x) \quad \dots \quad g_m(x))^T \quad \dots \quad$ (A2)

$y$ is another column vector with appropriate dimensions.

$A$ and $Q$ are constant square matrices, with appropriate dimensions.

1. $\frac{\partial}{\partial x}(Ax)=A^T$
2. $\frac{\partial}{\partial x}(x^T A)=A$
3. $\frac{\partial}{\partial x}(x^T Ay) = Ay$
4. $\frac{\partial}{\partial x}(y^T Ax) = A^T y$
5. $\frac{\partial}{\partial x}(x^T Ax) = (A+A^T)x$

6. $\frac{\partial}{\partial x}[f^T(x)g(x)] = \left[\frac{\partial}{\partial x}f(x)\right]^T g(x) + \left[\frac{\partial}{\partial x}g(x)\right]^T f(x)$
7. $\frac{\partial}{\partial x}(x^T x) = 2x$
8. $\frac{\partial}{\partial x}(x^T Q x) = 2Qx$ ; where $Q = Q^T$