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The Inverse Problem of the Atmospheric Refraction

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Abstract

In the present paper, an analytical formula was established for the inverse problem of the atmospheric refraction. Numerical applications are also given with computational checks, which indicate the extreme accuracy of the formula. An application of the formula to study the efficiency of astronomical refractions formulae was also illustrated.

Keywords: Atmospheric refraction, spherical astronomy, inverse problems

1. Introduction

Atmospheric refraction was mentioned as early as the first century A.D. by Cleomedes independently by Ptolemy (discussed in his *Optics*), ca. A.D.150. The refraction is the bending of light while passing from transparent homogenous medium to another transparent homogenous medium whose density is different from the first medium. The starlight moves in straight line until it meets the outer surface of the atmosphere, then it suffers through its passage in the Earth's atmosphere series of refractions called *astronomical refraction*. As a result of refraction, the observed altitude of a celestial object is greater than its geometric altitude.

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The amount of refraction depends on the altitude of the object and on atmospheric conditions (Timofeyev and Vasil′ev 2008). The numerical formula for the *mean refraction* R for the standard conditions which are taken in practice to be: barometric pressure =760 mm Hg. and temperature = $10^o C.$, is given by one of the following formulae

$$R = 58.29\ 4''\ \tan\xi - 0.066\ 8''\ \tan^3\xi. , \tag{1}$$

$$R = 58.27\ 6''\ \tan\xi - 0.082\ 4''\ \tan^3\xi, \tag{2}$$

where ξ is the observed (apparent) zenith distance. The first formula is given by Smart (1980) while, the second one has been derived by Meeus (1999). Clearly, both formulae (1) and (2) are not valid when the observed zenith distance equal to 90^o. In addition, the formulae are insufficient when the zenith distance exceeds 75^o (Corbard et al. 2014). Atmospheric refraction plays important roles in many applications of spherical astronomy, of these as for example, in topo centric phenomena, such as the time of rising and setting of the Sun and Moon, and in the prediction of local circumstances of eclipses. Also for observational reductions, the effect of refraction on the equatorial coordinates of a star must be included. On the other hand, the inverse problem of the atmospheric refraction that is to determine the true zenith distance z (or the apparent zenith distance $\xi = z - R$) knowing the atmospheric refraction R is also important in many applications of spherical astronomy (Taff 1981,Meeus 1999) The present paper is devoted to establish analytical formula for inverse problem of the atmospheric refraction. Numerical applications are also given with computational checks which indicating the extreme accuracy of the formula. An application of the formula to study the efficiency of astronomical refractions formulae was also illustrated.

2. Analytical Solution of the Inverse Problem

The inverse problem of the atmospheric refraction is to find z knowing R. In this respect, we established the following theorem.

Theorem 1: If the value of the atmospheric refraction R (arc seconds)is known, then the true zenith distance z is given as*:*

$$z = R + \arctan\left[\frac{-s^2 + \sqrt{s^4 + 48.54368932\ s}}{2s}\right], \tag{3}$$

where

$$s = 2 - C^2(0.222222 + 0.0003725356\ R), \tag{4}$$

C is the continued fraction:

$$C = \cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \cfrac{n_3}{d_3 + \ddots}}}, \tag{5}$$

$$\mathrm{n}_1 = 1,\ n_i = \begin{cases} \dfrac{-\eta}{9}\dfrac{(3\,i-1)(3\,i\text{-}4)}{(2\,i\text{-}3)(2\,i\text{-}1)} & i \quad \text{odd} \quad > 1, \\ \dfrac{-\eta}{9}\dfrac{(3\,i-2)(3\,i\text{-}5)}{(2\,i\text{-}3)(2\,i\text{-}1)}. & i \quad \text{even} \quad \geq 2,. \end{cases} \qquad \mathrm{d}_\mathrm{j} = 1\ \forall \mathrm{j} \geq 1, \tag{6}$$

$$\eta = \frac{1}{2}(1 - b) = (1 + 0.001676412\ R). \tag{7}$$

Proof:

Equation (2) can be written as:

$$\tan^3\xi - p\tan\xi = q, \tag{8}$$

where

$$p = \frac{58.276''}{0.0824''} = 707.233097, \tag{9}$$

$$q = \frac{-R''}{0.0824''} = -12.13592233\, R. \tag{10}$$

Let

$$\tan\xi = \sqrt{\frac{p}{3}}\, y, \tag{11}$$

then

$$y^3 - 3y = 2b, \tag{12}$$

where

$$b = \frac{q}{2}\left(\frac{1}{3}p\right)^{-3/2} = -0.001676412\, R. \tag{13}$$

Let

$$y = \nu + \nu^{-1}, \tag{14}$$

then *y* could be represented as:

$$y = (b + \sqrt{b^2+1})^{1/3} + (b + \sqrt{b^2+1})^{-1/3}. \tag{15}$$

Since, $|b| < 1$ then we can write

$$b = \cos \ell x, \tag{16}$$

where ℓ any integer. Using Equation (16) into Equation (15), we get:

$$y = 2\cos\frac{\ell}{3}x, \tag{17}$$

consequently, Equation (12) yields

$$4\cos^3\frac{\ell}{3}x - 3\cos\frac{\ell}{3}x = \cos \ell x \ \forall \ell, \tag{18}$$

which is an appropriate identity for the cosine function for any integer ℓ. Equation (17) could be written as:

$$y = 2 - 4\left(\frac{\sin \ell x/6}{\sin \ell x/2}\right)^2 \sin^2 \ell x/2\,. \tag{19}$$

Since ℓ at our disposal let us take $\ell = 2$, then Equation (19) reduces to

$$y = 2 - \frac{2\ (1\text{-}b)}{9}\left(\frac{\sin x/3}{\sin x}\right)^2. \tag{20}$$

Using the continued expansion of the ratio in the brackets (Wall 1948), Equations (5) and (6) follow directly. Equation (7) obtained by using the value of b (Equation (13)).

Having obtained the solution (20), the smallest positive root ξ of Equation (12) is obtained from the solution of the equation

$$\xi^2\xi_1 + \xi_1^3\xi + q = 0\ ; \quad \xi_1 = y\sqrt{\frac{p}{3}}\,,$$

and hence the true zenith distance from $z = R + \xi$ as given from Equation (3).

3. Continued Fraction Evaluation

In fact, continued fraction expansions are generally far more efficient tools for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series.

3.1 Top- Down Continued Fraction Evaluation

There are several methods available for the evaluation of continued fraction. Traditionally, either the fraction was computed from the bottom up, or the numerator and denominator of the nth convergent were accumulated separately with three-term recurrence formulae. The drawback of the first method, obviously, has to decide far down the fraction to being in order to ensure convergence. The drawback to the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well-defined limit. Thus, it is clear that an algorithm that works from top down while avoiding numerical difficulties would be ideal from a programming standpoint. Gautschi (1967) proposed very concise algorithm to evaluate continued fraction from the top down and may be summarized as follows. If the continued fraction is written as in Equation (5), then initialize the following parameters

$$a_1 = 1,$$
$$b_1 = n_1 / d_1,$$
$$c_1 = n_1 / d_1$$

and iterate (k=1, 2,…) according to:

$$a_{k+1} = \frac{1}{1 + \left[\frac{n_{k+1}}{d_k d_{k+1}}\right] a_k},$$

$$b_{k+1} = \left[a_{k+1} - 1\right] b_k,$$

$$c_{k+1} = c_k + b_{k+1}.$$

In the limit, the c sequence converges to the value of the continued fraction.

4. Applications

4.1 Numerical Applications

Consider random values of the atmospheric refraction $R \in [1,180]$ seconds of arc, the applications of Theorem 1 and Gautschi's algorithm of Section 3 yields the true zenith distances as listed in Table 1.The computed values could be checked by the condition that

$$CH = y^3 - 3y - 2b \ , \tag{21}$$

such that, the smaller the value of CH the more accurate solution will be.

Table 1: The Inverse Problem of the Atmospheric Refraction

R”	z°	CH	R”	z°	CH
39.9755	34.4778	-8.88178e-16	47.2943	39.1006	-2.7756e-16
84.7417	55.5882	1.4988e-15	63.7008	47.6126	-8.32667e-16
154.556	69.576	1.9984e-15	130.447	66.1175	-3.10862e-15
53.6412	42.6778	2.77555e-17	4.27549	4.19727	6.59195e-17
153.961	69.5018	2.33147e-15	119.489	64.1702	-1.11022e-16
9.43967	9.20393	6.8695e-16	118.403	63.9615	-1.22125e-15
62.5668	47.0977	1.58207e-15	42.0237	35.8276	1.11022e-16
126.552	65.45569	-1.66533e-16	16.7368	16.0305	4.23273e-16
143.874	68.1646	-3.88578e-16	177.756	72.1276	-1.11022e-16
86.5408	56.1513	1.11022e-15	83.0376	55.0396	4.44089e-16
60.6897	46..2232	-2.13718e-15	121.277	64.5074	5.55112e-16
22.3535	20.9960	1.11022e-15	86.9758	56.2851	1.38778e-15
150.218	69.0236	7.77156e-16	102.129	60.4269	1.11022e-16
52.1028	41.8457	2.77556e-16	87.864	56.5554	7.21645e-16
62.8364	47.2210	-1.05471e-15	30.826	27.8952	-6.93889e-17

4.2 Efficiency Study

Due to the extreme accuracy of the expression (3), we can use it to study the efficiency of the many astronomical refraction formulae (for example the formulae mentioned in Bennett's paper (Bennett 1982)) as follows:

1-Select one of the formulae, $R = F(\xi)(\text{say})$.

2- Compute the values of R_j for some given values of $\xi_j; \mathrm{j} = 1,2,\cdots,\mathrm{N}(\text{say})$.

3- Using these values of R's to compute ξ'_j from Equation (3) as

$\xi'_j = z_j - R_j;\ j = 1, 2, \ldots, N.$

4- Compute the absolute relative errors $\Delta_j = \dfrac{\left|\xi_j - \xi'_j\right|}{\xi'_j};\ j = 1, 2, \ldots, N$.

Now let us renormalize the concept of acceptable "astronomical refraction formula" We may define an acceptable set for astronomical formulae as:

$$W = \left\{ \overline{\Delta} = \sum_{j}^{N} \Delta_j \Big/ N : \overline{\Delta} \le Tol \right\}, \tag{22}$$

Where *Tol* is small number. In writing Equation (22), we do not mean to establish this particular definition of an acceptable set, as it is only intended to give the users some degree of concreteness to the general idea of acceptable formulae set. In concluding the present paper, an analytical formula was established for the inverse problem of the atmospheric refraction. Numerical applications are also given with computational checks which indicating the extreme accuracy of the formula. An application of the formula to study the efficiency of astronomical refractions formulae was also illustrated.

References

Battin, R. H., 1999: An Introduction to the Mathematics and Methods of Astrodynamics, Revised Edition, AIAA, Education Series, Reston, Virginia.

Bennett. G, G., 1982: The calculation of Astronomical Refraction Marine Navigation, Journal of the Institute for Navigation, Vol. 35, pages 255-259

Corbard, T., Morand, F., Laclare, F., Ikhlef, R. and Meftah, M., 2014: On the importance of astronomical refraction for modern Solar astrometric measurements, arXiv: 1304.0147v2 [astro-ph.SR].

Gautschi, W., 1967: Computational Aspects of Three-term Recurrence Relations, SIAM Review, Vol.9, No. 1, January.

Meeus, J., 1999: Astronomical Algorithms, 2nd edition, willmann-Bell.

Smart, W. M., 1980: Text Book on Spherical Astronomy, Cambridge University Press.

Taff, L. G., 1981: Computational Spherical Astronomy, John Wiley Sons, New York.

Wall, H.S., 1948: Analytic Theory of Continued Fractions, D.VanNostrand Company, Inc. London.

Yu. M. Timofeyev, A. V. Vasil′ev, 2008: Theoretical Fundamentals of Atmospheric Optics, Cambridge Int. Science Publishing.